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An Eigenvector Interpretation of an Array's Bearing-Response Pattern

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ABSTRACT

At a single frequency an array's bearing-response pattern is expressed as a weighted sum of terms involving a steering matrix and the eigenvalues and eigenvectors of the array's spectral density matrix. This point of view is used to gain an understanding of the beam patterns generated by the Adaptive Search and Track Array (ASTA) processor.

ADMINISTRATIVE INFORMATION

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INTRODUCTION

Conventional array bearing-response patterns are derived in terms of the eigenvector decomposition of the sampled acoustic field's spectral density matrix. The Adaptive Search and Track Array (ASTA) processing technique proposed by Owsley[1] derives filters for a multisensor array from the components of the eigenvector corresponding to the largest eigenvalue of the spectral density matrix. The relationship between beam patterns resulting from this choice of filters and bearing-response patterns due to conventional time-delay beamforming provides a basis for a better understanding of both processing techniques.

The bearing-response pattern of an n-element array of hydrophones can be expressed in terms of the spectral density matrix of the received data, which, in turn, can be written as a weighted sum of outer products. Each outer product uses an eigenvector of the spectral density matrix and is weighted by the corresponding eigenvalue.

In an acoustic field having equal power at all hydrophones and no spatial coherence, all eigenvalues of the spectral density matrix are equal. However, as targets appear, the spatial coherence of the received data increases, and the eigenvalues, in general, will no longer be equal. Thus, the information contained in the bearing-response pattern becomes concentrated in the terms of its eigenvector expansion corresponding to the larger eigenvalues. The equation for a beam pattern of the ASTA processor is the dominant term in the eigenvector expansion of the array's conventional bearing-response pattern.

DERIVATION OF BEARING-RESPONSE EXPANSION

Consider the conventional beamformer of Fig. 1.

[1] N. L. Owsley, "An Adaptive Search and Track Array (ASTA)," NUSL Technical Memorandum No. 2242-166-69, 7 July 1969.

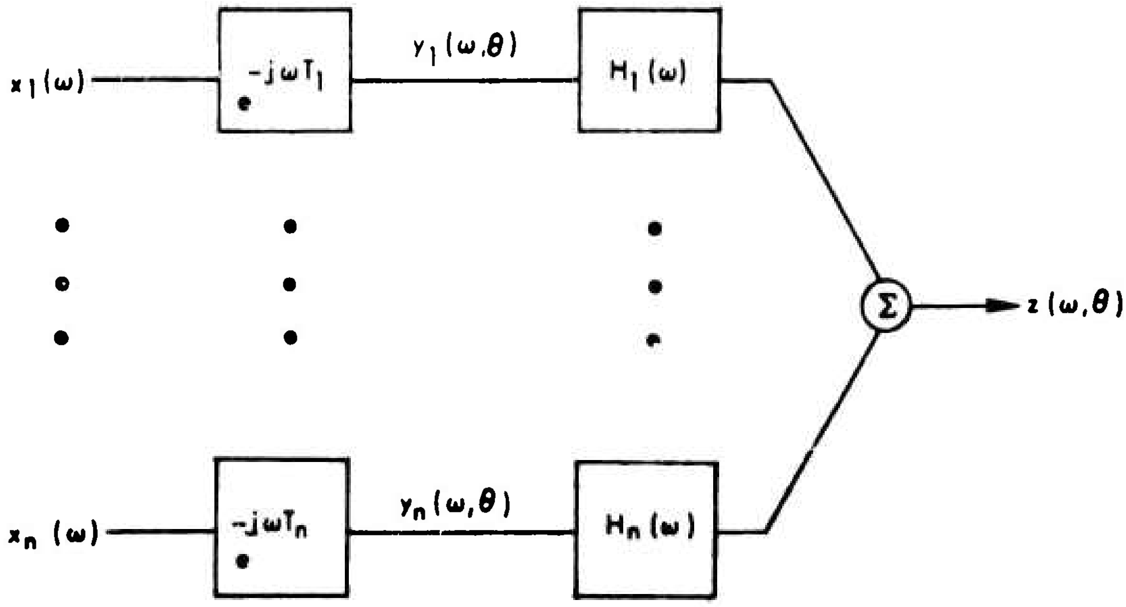

Fig. 1 - Conventional Beamformer

Let the $\{x_i(\omega)\}$, $i = 1, 2, \ldots, n$ be single frequency, complex random processes at radian frequency ω, and let the $\{e^{-j\omega T_i}\}$ be the phase shifts required to steer the array to some direction θ. The $\{H_i(\omega)\}$ are transfer functions of linear filters that may be used to "shade" the array.

We define the array's bearing-response pattern as the beamformer output power versus the steering direction θ. In general, this pattern differs from the array's farfield beam pattern since the latter is obtained by fixing the steering direction to some angle θ_o and then plotting the beamformer output power versus the angle of a single, farfield source.

Let us define vectors $x(\omega)$, $y(\omega, \theta)$, and $H(\omega)$ as

$$x(\omega) = \begin{bmatrix} x_1(\omega) \\ \cdot \\ \cdot \\ \cdot \\ x_n(\omega) \end{bmatrix} \qquad y(\omega,\theta) = \begin{bmatrix} y_1(\omega,\theta) \\ \cdot \\ \cdot \\ \cdot \\ y_n(\omega,\theta) \end{bmatrix} \qquad H(\omega) = \begin{bmatrix} H_1(\omega) \\ \cdot \\ \cdot \\ \cdot \\ H_n(\omega) \end{bmatrix}$$

and a delay matrix $D(\omega, \theta)$ as

$$D(\omega, \theta) = \begin{bmatrix} e^{-j\omega T_1} & & & & 0 \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & e^{-j\omega T_n} \end{bmatrix} \qquad (1)$$

Then,

$$y(\omega, \theta) = D(\omega, \theta)\, x(\omega) \quad .$$

The beamformer output is now

$$z(\omega, \theta) = H^T(\omega)\, y(\omega, \theta) = y^T(\omega, \theta)\, H(\omega) \quad . \qquad (2)$$

The bearing-response function is defined by

$$B(\omega, \theta) = E\,|\,z(\omega, \theta)\,|^2 = E\left[z^*(\omega, \theta)\, z(\omega, \theta)\right] \quad ,$$

where E denotes expected value. Substitution of Eq. (2) into Eq. (3) yields

$$\begin{aligned} B(\omega, \theta) &= E\left[H^{*T}(\omega)\, y^*(\omega, \theta)\, y^T(\omega, \theta)\, H(\omega)\right] \quad . \\ &= E\left[H^{*T}(\omega)\, D^*(\omega, \theta)\, x^*(\omega)\, x^T(\omega)\, D^T(\omega, \theta)\, H(\omega)\right] \\ &= H^{*T}(\omega)\, D^*(\omega, \theta) \left\{ E\left[x^*(\omega)\, x^T(\omega)\right]\right\} \left[\, D(\omega, \theta)\, H(\omega)\right] \quad . \end{aligned} \qquad (4)$$

But the bracketed quantity is just the spectral density matrix of the inputs; i.e.,

$$S_{xx}(\omega) = E\left[x^*(\omega)\, x^T(\omega)\right] = \begin{bmatrix} S_{11}(\omega) & \cdot & \cdot & \cdot & S_{1n}(\omega) \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ S_{n1}(\omega) & \cdot & \cdot & \cdot & S_{nn}(\omega) \end{bmatrix} \quad , \qquad (5)$$

where $S_{ij}(\omega)$ is the cross-spectral density function for the ith and jth inputs. Thus, we can write

$$B(\omega, \theta) = H^{*T}(\omega)\, D^*(\omega, \theta)\, S_{xx}(\omega)\, D(\omega, \theta)\, H(\omega) \quad . \qquad (6)$$

We now take advantage of the following expansion for $S_{xx}(\omega)$ (See the Appendix):

$$S_{xx}(\omega) = \sum_{K=1}^{n} \lambda_K(\omega)\, m_K(\omega)\, m_K^{*T}(\omega) \quad . \qquad (7)$$

The $\{\lambda_K(\omega)\}$ are the eigenvalues of $S_{xx}(\omega)$, and the $\{m_K(\omega)\}$ are the corresponding normalized eigenvectors. Substituting Eq. (7) into Eq. (6) yields

$$B(\omega,\theta) = \sum_{K=1}^{n} \lambda_K(\omega) \left[H^{*T}(\omega)\, D^{*}(\omega,\theta)\, m_K(\omega)\right] \left[m_K^{*T}(\omega)\, D(\omega,\theta)\, H(\omega)\right] . \quad (8)$$

But

$$H^{*T}(\omega)\, D^{*}(\omega,\theta)\, m_K(\omega) = \left[m_K^{*T}(\omega)\, D(\omega,\theta)\, H(\omega)\right]^{*T} = \text{scalar};$$

therefore,

$$B(\omega,\theta) = \sum_{K=1}^{n} \lambda_K(\omega) \left| H^{*T}(\omega)\, D^{*}(\omega,\theta)\, m_K(\omega)\right|^2 . \quad (9)$$

Thus, we have expressed the array's single frequency, bearing-response pattern as a weighted sum of terms, each of which depends only on a single eigenvector; the weights in the sum are the corresponding eigenvalues.

DISCUSSION OF ASTA BEAM PATTERNS

ASTA has the following structure:

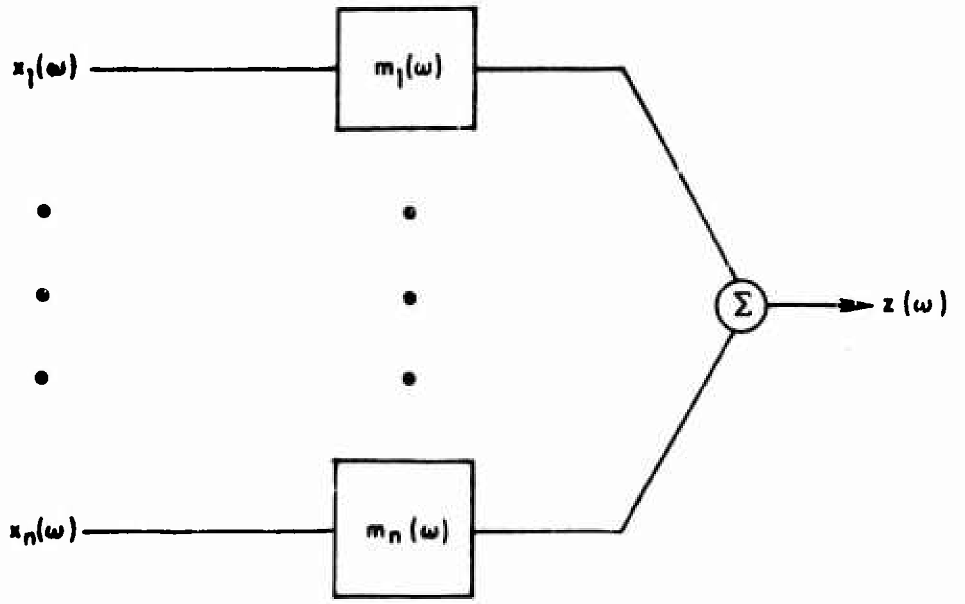

Fig. 2 - ASTA Processor

Notice that no steering delays are present. The filters $\{m_K(\omega)\}$ are determined from the eigenvector associated with the largest eigenvalue of the array's spectral density matrix; i. e.,

$$S_{xx}(\omega)\, m(\omega) = \lambda_{max}\, m(\omega) \quad ,$$

where

$$m(\omega) = \begin{bmatrix} m_1(\omega) \\ \cdot \\ \cdot \\ \cdot \\ m_n(\omega) \end{bmatrix} .$$

It can be shown [2] that with these filters the system's output power is maximized under the constraint that

$$m^{*T}(\omega)\, m(\omega) = \text{constant} \quad .$$

From simulation results, Owsley [3] has found that a beam pattern for ASTA will have lobes in the direction of the targets. Since the bearing-response pattern of a conventional beamformer also has lobes in the direction of the targets, one might wonder if there is any connection between the two patterns. The answer is yes, as will now be shown.

To compute a beam pattern for ASTA, let $T(\omega)$ be a single frequency, complex output of a farfield source in some direction θ. The vector of received signals is then

$$x(\omega) = T(\omega)\, D^{-1}(\omega, \theta)\, 1 = T(\omega)\, D^{*}(\omega, \theta)\, 1 \quad ,$$

where $D(\omega,\theta)$ is as defined above (Eq. (1)), and

$$1 = \begin{bmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} .$$

The beampattern $\beta(\omega,\theta)$ is the output power as a function of θ where θ is the direction of the farfield source; i. e.,

$$\beta(\omega, \theta) = E\,|\, z(\omega, \theta)\,|^2 \quad .$$

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[2] J. N. Franklin, Matrix Theory, Prentice Hall, Inc., Englewood Cliffs, N. Y., 1968, p. 142.
[3] See Footnote 1.

But

$$z(\omega, \theta) = x^T(\omega)\, m(\omega) = T(\omega)\, 1^T D^*(\omega,\theta)\, m(\omega) \quad . \tag{10}$$

As a result,

$$\beta(\omega, \theta) = \left\{ E\, |T(\omega)|^2 \right\} \left| 1^T D^*(\omega, \theta)\, m(\omega) \right|^2 \quad . \tag{11}$$

Comparing Eqs. (9) and (11), we see that for the case of uniform shading of the conventional processor, i.e., $H(\omega) = H^*(\omega) = 1$, $\beta(\omega, \theta)$ equals (to within a scale factor) one term of $B(\omega, \theta)$.

If the acoustic field is of uniform power at all hydrophones and possesses no spatially coherent component, $S_{xx}(\omega)$ will be proportional to the identity matrix. In this case all eigenvalues of $S_{xx}(\omega)$ will be equal, and, therefore, each term of $\beta(\omega, \theta)$ is weighted by the same amount. A target, however, will add a spatially coherent component to the acoustic field and cause $S_{xx}(\omega)$ to deviate from a diagonal matrix. The eigenvalues of $S_{xx}(\omega)$ will no longer be equal in general; in fact, certain terms of $B(\omega, \theta)$ will dominate over others. The ASTA processor provides us with the term corresponding to λ_{max} , i.e., the most heavily weighted component of $B(\omega, \theta)$.

SUMMARY

It has been shown that at a single frequency, an array's bearing-response pattern can be expressed as a weighted sum of terms involving a steering matrix and an eigenvector of the array's spectral matrix; the weights in the sum are the corresponding eigenvalues. This result has been used to show that a beam pattern for the ASTA processor represents the dominant component of the conventional beamformer's bearing-response pattern. The results obtained here are generalizable to the broadband case.

Appendix

DERIVATION OF EIGENVECTOR EXPANSION FOR SPECTRAL DENSITY MATRIX

Let $S_{xx}(\omega)$ be a cross-spectral density matrix. Since $S_{xx}(\omega)$ is nonnegative definite, it can be diagonalized by the following unitary transformation:

$$\lambda(\omega) = M^{*T}(\omega)\, S_{xx}(\omega)\, M(\omega) = \begin{bmatrix} \lambda_1(\omega) & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & \lambda_n(\omega) \end{bmatrix} ,$$

where the $\{\lambda_i(\omega)\}$ are the eigenvalues of $S_{xx}(\omega)$ and $M(\omega)$ = modal matrix of $S_{xx}(\omega)$; i.e., $M(\omega) = [m_1(\omega), \ldots, m_n(\omega)]$, where the $\{m_i(\omega)\}$ are orthonormal eigenvectors of $S_{xx}(\omega)$. Solving for $S_{xx}(\omega)$ gives

$$S_{xx}(\omega) = M(\omega)\, \lambda(\omega)\, M^{*T}(\omega) .$$

The (i, k) element of $M(\omega)\,\lambda(\omega)$ is

$$\mu_{iK}(\omega) = \sum_{q=1}^{n} \left(m_q(\omega)\right)_i \lambda_K(\omega)\, \delta_{Kq} ,$$

where $(m_q(\omega))_i$ = ith component of $m_q(\omega)$ and

$$\delta_{Kq} = \begin{cases} 1 , & K = q \\ 0 , & K \neq q \end{cases} .$$

The (i, j) element of $M(\omega)\,\lambda(\omega)\,M^{*T}(\omega)$ is now

$$\begin{aligned} \sigma_{ij}(\omega) &= \sum_{K=1}^{n} \mu_{iK}(\omega)\left(m_K^*(\omega)\right)_j \\ &= \sum_{K=1}^{n} \sum_{q=1}^{n} \left(m_q(\omega)\right)_i \lambda_K(\omega)\, \delta_{Kq} \left(m_K^*(\omega)\right)_j \\ &= \sum_{K=1}^{n} \left(m_K(\omega)\right)_i \lambda_K(\omega) \left(m_K^*(\omega)\right)_j . \end{aligned}$$

Thus, we have

$$S_{xx}(\omega) = \{\sigma_{ij}(\omega)\} = \sum_{K=1}^{n} \lambda_K(\omega)\, m_K(\omega)\, m_K^{*T}(\omega) .$$

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